# New Classes of Priors based on Stochastic Orders: Theory and Applications in Reliability

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# BAYESIAN ROBUSTNESS - BERGER, 1985

- $X \sim \mathcal{N}(\theta, 1)$
- Expert's opinion on prior $P$: median at $0$, quartiles at $\pm 1$, symmetric and unimodal
- $\Rightarrow$ Possible priors include Cauchy $\mathcal{C}(0, 1)$ and Gaussian $\mathcal{N}(0, 2.19)$
- Interest in posterior mean $\mu^C(x)$ or $\mu^N(x)$

| $x$ | 0 | 1 | 2 | 4.5 | 10 |
|---|---|---|---|---|---|
| $\mu^C(x)$ | 0 | 0.52 | 1.27 | 4.09 | 9.80 |
| $\mu^N(x)$ | 0 | 0.69 | 1.37 | 3.09 | 6.87 |

- Decision strongly dependent on the choice of the prior for large $x$
- $\Rightarrow$ Choice of a class Γ of priors
- $\Rightarrow$ Robustness measure: range of posterior quantity of interest

# STOCHASTIC ORDERS

- *Usual* stochastic order
  - $X$ and $Y$ r.v.'s with d.f.'s $F_X$ and $F_Y$ s.t. $F_X(t) \geq F_Y(t), \quad \forall t \in \mathbb{R}$
  - $\Rightarrow X \leq_{\text{st}} Y$, i.e. $X$ is said to be *smaller than* $Y$ *in the usual stochastic order*
  - $X \leq_{\text{st}} Y \Leftrightarrow E[g(X)] \leq E[g(Y)]$ holds for all increasing functions $g$ for which the expectations exist

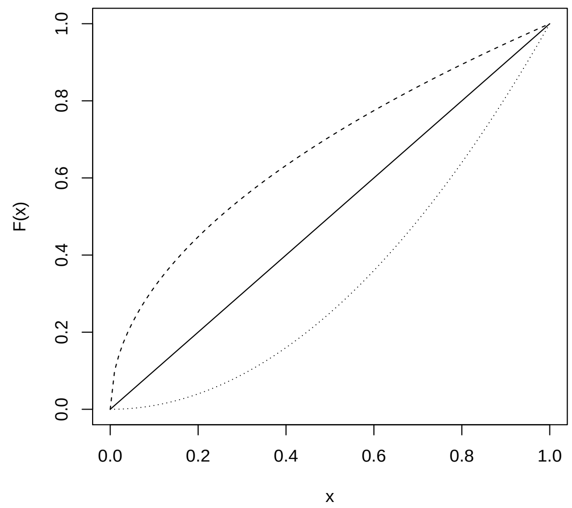
- Likelihood ratio order
  - $X$ and $Y$ be (discrete) absolutely continuous r.v.'s with d.f.'s $F_X$ and $F_Y$ and (discrete) densities $f_X$ and $f_Y$ s.t. $\dfrac{f_Y(t)}{f_X(t)}$ increases over the union of the supports of $X$ and $Y$ (here $a/0$ is taken to be equal to $\infty$ whenever $a > 0$)
  - $\Rightarrow X \leq_{\text{lr}} Y$, i.e. $X$ is said to be *smaller than* $Y$ *in the likelihood ratio order*

- $X \leq_{lr} Y \Rightarrow X \leq_{st} Y$

# DISTORTION FUNCTIONS

- $X$ r.v. with d.f. $F_X$
- $h$ distortion function
  - non-decreasing continuous function $h : [0, 1] \to [0, 1]$
  - s.t. $h(0) = 0$ and $h(1) = 1$
- Given $h$, d.f. modified by $F_h(x) = h \circ F(x) = h[F(x)]$
- $\Rightarrow X_h$ distorted r.v. with d.f. $F_h(x)$
- **Lemma.**
  - $\pi$ prior distribution (absolutely continuous or discrete) with d.f. $F_\pi$
  - $h$ convex distortion function in $[0, 1] \Rightarrow \pi \leq_{lr} \pi_h$
  - $h$ concave distortion function in $[0, 1] \Rightarrow \pi \geq_{lr} \pi_h$

# CONCAVE AND CONVEX DISTORTION FUNCTIONS




- Solid: $F_{\pi}(\theta) = \theta$
- Dashed: $F_{\pi_{h_1}}(\theta) = \sqrt{\theta}$ (concave distortion) $\Rightarrow$ decreasing l.r. $= \dfrac{1}{2\sqrt{\theta}} \Rightarrow \pi \geq_{lr} \pi_{h_1}$
- Dotted: $F_{\pi_{h_2}}(\theta) = \theta^2$ (convex distortion) $\Rightarrow$ increasing l.r. $= 2\theta \Rightarrow \pi \leq_{lr} \pi_{h_2}$

# DISTORTED BAND OF PRIORS

- Uncertainty on prior $\pi$ through concave ($h_1$) and convex ($h_2$) distortion functions
- $\Rightarrow$ distorted distributions $\pi_{h_1}$ and $\pi_{h_2}$ s.t. $\pi_{h_1} \leq_{lr} \pi \leq_{lr} \pi_{h_2}$
- **Definition.** Distorted band $\Gamma_{h_1,h_2,\pi}$ s.t. $\Gamma_{h_1,h_2,\pi} = \{\pi' : \pi_{h_1} \leq_{lr} \pi' \leq_{lr} \pi_{h_2}\}$
- $\pi \in \Gamma_{h_1,h_2,\pi} \Rightarrow$ distorted band as a particular "neighborhood" band of $\pi$, with lower and upper bounds given by distorted distributions
- $X \leq_{lr} Y \Rightarrow X \leq_{st} Y \Rightarrow$ distorted band subclass of distribution band class, i.e.

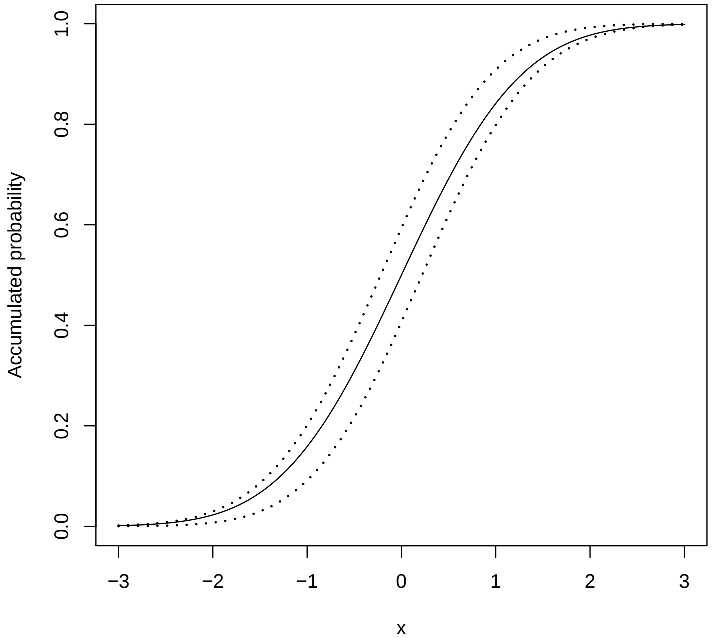
$$\begin{aligned} \Gamma_{h_1,h_2,\pi} &\subseteq \left\{\pi' : \pi_{h_1} \leq_{st} \pi' \leq_{st} \pi_{h_2}\right\}, \\ &= \left\{\pi' : F_{\pi_{h_1}}(\theta) \geq F_{\pi'}(\theta) \geq F_{\pi_{h_2}}(\theta), \forall \theta \in \Theta\right\} \end{aligned}$$

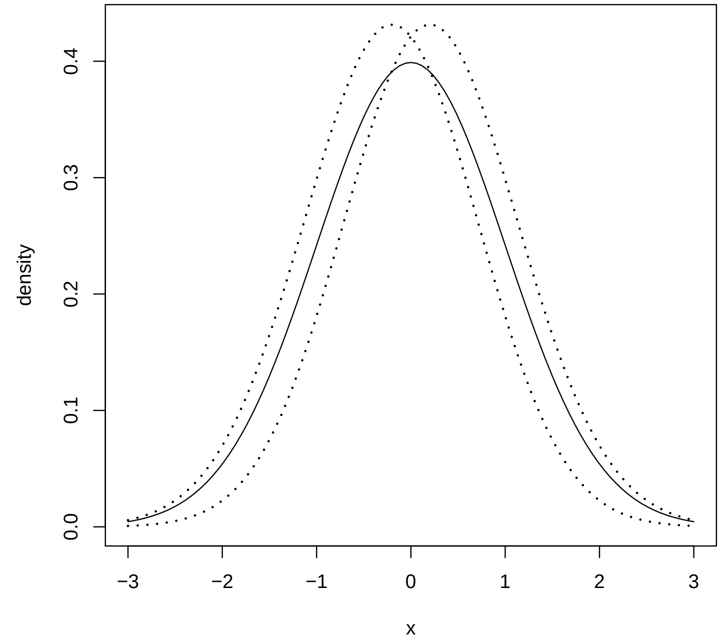
- Note that likelihood ratio order does not apply, in general, when comparing two priors $\pi'_1$ and $\pi'_2$ in $\Gamma_{h_1,h_2,\pi}$, since each of them is just ordered w.r.t. $\pi_{h_1}$ and $\pi_{h_2}$
- We are unable to "identify" all the priors in the class

# CHOICES OF DISTORTION FUNCTIONS

- $h_1(x) = 1 - (1-x)^\alpha$ and $h_2(x) = x^\alpha, \quad \forall \alpha > 1$
  - Useful to represent uncertainty in the tails of the prior
  - $\alpha = n \in \mathbb{N} \Rightarrow F_{\pi_{h_1}}(\theta) = 1 - (1 - F_\pi(\theta))^n$ and $F_{\pi_{h_2}}(\theta) = (F_\pi(\theta))^n$
  - $\Rightarrow$ d.f.'s of min and max of i.i.d. random sample of size $n$ from baseline prior $\pi$
- Skewed distributions
  - Absolutely continuous, symmetric around 0 prior with density $\pi(\theta)$ and d.f. $F_\pi(\theta)$
  - $\Rightarrow$ skew-$\pi$ with parameter $\alpha$ with density $\pi_\alpha(\theta) = 2\pi(\theta)F_\pi(\alpha\theta)$
  - Distribution: right skewed if $\alpha > 0$ and left skewed if $\alpha < 0$
  - Easy to show $\pi \leq_{lr} \pi_\alpha$ for all $\alpha > 0$ and $\pi_\alpha \leq_{lr} \pi$ for all $\alpha < 0$

# CHOICES OF DISTORTION FUNCTIONS




- $\pi \sim N(0,1)$ prior with standard normal d.f. $\Phi_Z$
- Distorted d.f.'s $F_{\pi_{h_1}}(\theta) = 1 - (1 - \Phi_Z(\theta))^{1.3}$ and $F_{\pi_{h_2}}(\theta) = (\Phi_Z(\theta))^{1.3}$

# POSTERIOR BAND

- Spizzichino (2001): given two priors $\pi_1$ and $\pi_2$ s.t. $\pi_1 \leq_{lr} \pi_2$

  $\Rightarrow$ posteriors s.t. $\pi_{1x} \leq_{lr} \pi_{2x}$

- **Proposition.** $\pi$ prior and $\Gamma_{h_1,h_2,\pi}$ distorted band around $\pi$ based on $h_1$ and $h_2$

  $\Rightarrow \pi_{h_1,x} \leq_{lr} \pi'_x \leq_{lr} \pi_{h_2,x} \ \forall \pi' \in \Gamma_{h_1,h_2,\pi}$

- Posterior of lower and upper bound distributions of the distribution band $\Rightarrow$ lower and upper bounds in the $\leq_{lr}$ order sense for $\Gamma_x$, family of posterior distributions

- $\Rightarrow$ $\Gamma_x$ still distortion band of a posterior for some concave and convex functions

- *Closure* property very uncommon among classes of priors

  $\Rightarrow$ dealing with priors or posteriors is the same

# METRICS TO MEASURE UNCERTAINTY

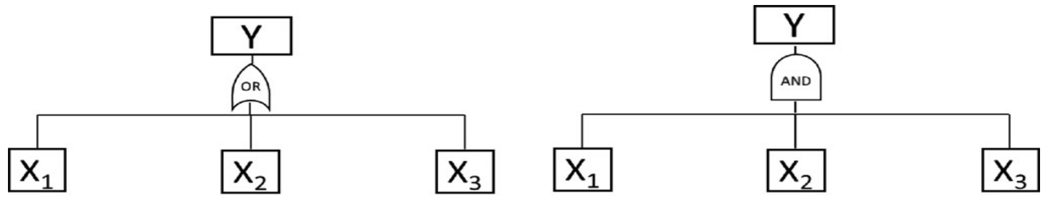
- Interest in probability metrics to evaluate how a prior belief differs from its distorted version and how the corresponding posterior distributions differ

- Mathematical tractability (but not only!) $\Rightarrow$ Kolmogorov and Kantorovich metrics

- R.v.'s $X$ and $Y$ with d.f.'s $F_X$ and $F_Y$

- Kolmogorov metric $K(X, Y)$
  - $K(X, Y) = \sup_{x \in \mathbf{R}} |F_X(x) - F_Y(x)| = |p_0 - h(p_0)|$, with $p_0$ s.t. $h'(p_0) = 1$
  - $h_1(x) = 1 - (1 - x)^\alpha$ and $h_2(x) = x^\alpha, \quad \forall \alpha > 1$
  - $\Rightarrow K(\pi, \pi_{h_1}) = K(\pi, \pi_{h_2}) = \dfrac{\alpha - 1}{\sqrt[\alpha-1]{\alpha^\alpha}} = 0.067$ for $\alpha = 1.2$

# FAULT TREE ANALYSIS




(a) (b)

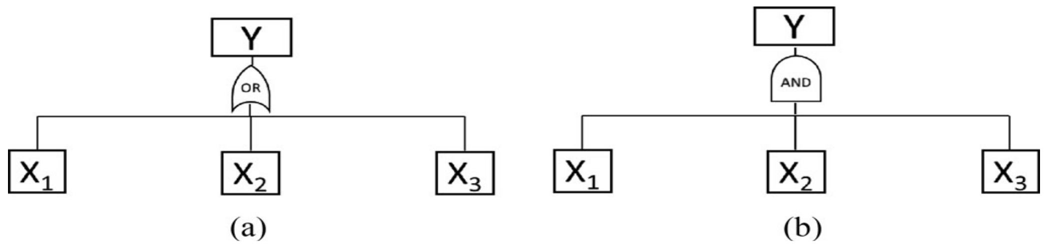
Two simple fault trees (a) using an OR operation and (b) using an AND operation

- Fault Tree Analysis (FTA) used to quantify the probability of occurrence of an undesirable event (Top Event, here $Y$)
- Fault trees (FTs) constructed in a top-down fashion from Top Event to their causes, represented by intermediate and elementary (here $X_1, X_2, X_3$) events in the tree
- Events usually Bernoulli trials $\Rightarrow$ either happen or not
- Events usually statistically independent
- Relationship between events and causes represented using logical gates, most commonly AND and OR gates

# FAULT TREE ANALYSIS

- FTA often used to evaluate risk in large, safety critical systems but has limitations due to its static structure
- Bayesian approaches proposed as a superior alternative to it, however, this involves prior elicitation, which is not straightforward
- Priors typically provided by Beta distributions on occurrence probability of elementary events
- Minor misspecification of priors for elementary events can result in a significant prior misspecification for the top event
- $\Rightarrow$ need for a robustness approach for FTA which can quantify the effects of prior misspecification on the posterior analysis
- $\Rightarrow$ first Bayesian robustness approach specifically developed for FTA

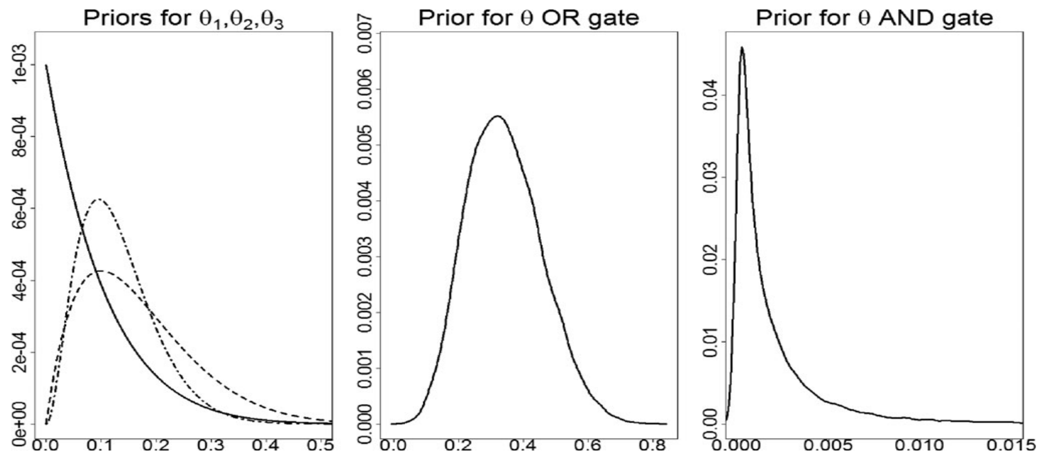
# FAULT TREE ANALYSIS




- $X_i \sim$ Bernoulli($\theta_i$), with $\theta_i = P(X_i = 1)$, $i = 1, 2, 3$,
- $Y \sim$ Bernoulli($\theta$), with $\theta = P(Y = 1)$
- $\Rightarrow \theta = 1 - \prod_{i=1}^{3}(1 - \theta_i)$ for [a] and $\theta = \prod_{i=1}^{3} \theta_i$ for [b]
- $\Rightarrow \pi(\theta = \tau) = \int_{\Omega_\tau^O} \pi_1(\tau_1)\pi_2(\tau_2)\pi_3(\tau_3)\, d\tau_1 d\tau_2 d\tau_3$ s.t.
  $\Omega_\tau^O = \{\tau_i \in [0, 1],\ i = 1, 2, 3 : 1 - \prod_{i=1}^{3}(1 - \tau_i) = \tau\}$ for the OR operation
- $\Rightarrow \pi(\theta = \tau) = \int_{\Omega_\tau^A} \pi_1(\tau_1)\pi_2(\tau_2)\pi_3(\tau_3)\, d\tau_1 d\tau_2 d\tau_3$ s.t.
  $\Omega_\tau^A = \{\tau_i \in [0, 1],\ i = 1, 2, 3 : \prod_{i=1}^{3} \tau_i = \tau\}$ for the AND operation

# FAULT TREE ANALYSIS

$\theta_1 \sim Beta(1, 10)$, $\theta_2 \sim Beta(2, 10)$ and $\theta_3 \sim Beta(3, 20)$

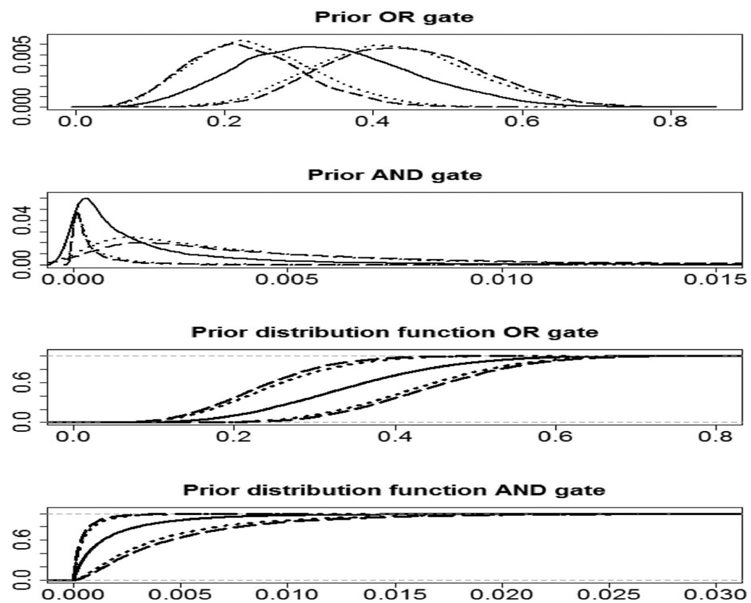



- **Left:** Priors for $\theta_1$ (*continuous*), $\theta_2$ (*dashed*) and $\theta_3$ (*dot-dashed*)
- **Centre:** Prior distribution for $\theta$ for the fault tree [a]
- **Right:** Prior distribution for $\theta$ for fault tree [b]

# FAULT TREE ANALYSIS

- Distorted band class $\Gamma_{\alpha_1}$ with same $\alpha_1$
  - Lower bound: $F_{h_1,\alpha_1}(x) = 1 - [1 - F(x)]^{\alpha_1}$
  - Upper bound: $F_{h_2,\alpha_1}(x) = [F(x)]^{\alpha_1}$
- $1 \leq \alpha_1 \leq \alpha_2 \Rightarrow \Gamma_{\alpha_1} \subset \Gamma_{\alpha_2}$
- $\Gamma_\alpha \to F(x)$ as $\alpha \downarrow 1$
- $\alpha_1$ chosen so that Kolmogorov distance does not exceed a threshold
- Algorithm for generating upper (lower) density in band for Top Event: $\theta_i$'s generated by upper (lower) density in elementary events
- Examples: spacecraft re-entry example and feeding control system

# FAULT TREE ANALYSIS




- *Continuous line:* starting prior for probability $\theta$ of Top Event $Y$
- *Long-dashed line:* distorted band using $\alpha = \mathbf{2}$
- *Dotted line:* distorted band using $\alpha = \mathbf{1.8}$
- Similar results for posterior distributions

# EXTENSION TO MULTIVARIATE CASE

- Prior $\pi(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^n$

- Weight function $\omega : \mathbb{R}^n \mapsto \mathbb{R}^+$ s.t. $0 < E^{\boldsymbol{\pi}}\left[\omega(\boldsymbol{\theta})\right] < \infty$

- Key idea: $\boldsymbol{\pi}_\omega(\boldsymbol{\theta}) = \dfrac{\omega(\boldsymbol{\theta})}{E^{\boldsymbol{\pi}}\left[\omega(\boldsymbol{\theta})\right]}\pi(\boldsymbol{\theta}), \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^n$

- For $n = 1$, distorted density function as weighted function:
  $f_{\pi_h}(\theta) = h^{'}(F_\pi(\theta))\pi(\theta) = w(\theta)\pi(\theta)$
  for absolutely continuous prior ($F_\pi(\theta)$ and $\pi(\theta)$) and differentiable distortion $h$

- $w(\theta) = h^{'}(F_\pi(\theta)) \Rightarrow$ weight depends on $F_\pi$

- Absolutely continuous r.v.'s $\Rightarrow$ weighted distributions more general than distorted ones

- Convex (concave) distortion $h \Leftrightarrow$ weight $w$ increasing (decreasing)

# EXTENSION TO MULTIVARIATE CASE

- $x$ and $y$ real numbers $\Rightarrow x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$
- $\mathbf{x}$ and $\mathbf{y}$ real vectors $\Rightarrow \mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ max and min componentwise
- $\mathbf{U} \subseteq \mathbb{R}^\mathbf{n}$ upper (lower) set if $\mathbf{y} \in \mathbf{U}$ whenever $\mathbf{y} \geq (\leq)\mathbf{x}$ and $\mathbf{x} \in \mathbf{U}$
- Increasing and decreasing used in a wide sense, i.e.,
  $g : \mathbb{R}^n \mapsto \mathbb{R}$ increasing (decreasing) if $g(\mathbf{x}) \leq (\geq) g(\mathbf{y})$ for all $\mathbf{x} \leq \mathbf{y}$
- $\mathbf{X}$ and $\mathbf{Y}$ $n$-dimensional r.v.'s with cdf's $F$ and $G$, respectively
- $\mathbf{X}$ smaller than $\mathbf{Y}$ in the usual multivariate stochastic order ($\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y}$) if
  $P\{\mathbf{X} \in \mathbf{U}\} \leq \mathbf{P}\{\mathbf{Y} \in \mathbf{U}\},$ for all upper sets $\mathbf{U} \subseteq \mathbb{R}^\mathbf{n}$
- $\mathbf{X}$ less likely than $\mathbf{Y}$ to take on large values, i.e. values in any upper set $\mathbf{U}$
- $\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y} \Rightarrow \mathbf{F}(\mathbf{x}) \geq \mathbf{G}(\mathbf{x}),\ \forall \mathbf{x} \in \mathbb{R}^\mathbf{n}$ (vice versa true only for $n = 1$)
- $\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y} \Leftrightarrow \mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$ for all increasing $\phi$ on $\mathbb{R}^n$ s.t. expectations exist

# EXTENSION TO MULTIVARIATE CASE

- $\mathbf{X}$ and $\mathbf{Y}$ two r.v.'s with pdf's $f$ and $g$, respectively
- $\mathbf{X}$ smaller than $\mathbf{Y}$ in the multivariate likelihood ratio stochastic order ($\mathbf{X} \leq_{\mathbf{lr}} \mathbf{Y}$) if $f(\mathbf{x})\mathbf{g}(\mathbf{y}) \leq \mathbf{f}(\mathbf{x} \wedge \mathbf{y})\mathbf{g}(\mathbf{x} \vee \mathbf{y})$, for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$
- $\mathbf{X} \leq_{\mathbf{lr}} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{\mathbf{st}} \mathbf{Y}$
- $l : \mathbb{R}^n \mapsto \mathbb{R}^+$, $n \geq 2$, multivariate totally positive of order 2 ($MTP_2$) if $l(\mathbf{x})\mathbf{l}(\mathbf{y}) \leq \mathbf{l}(\mathbf{x} \wedge \mathbf{y})\mathbf{l}(\mathbf{x} \vee \mathbf{y}), \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$
- $n$-dimensional r.v. $\mathbf{X}$ with pdf $f$ said $MTP_2$ if $f$ is $MTP_2$ or, equivalently, $\mathbf{X} \leq_{\mathbf{lr}} \mathbf{X}$
- Product of $MTP_2$ functions is still $MTP_2$
- $l(\mathbf{x}) = \prod_{i=1}^{n} g_i(x_i)$, with univariate non-negative $g_i$, $i = 1, \ldots, n$, $\Rightarrow l(\mathbf{x}) MTP_2$
- $g_i(x_i)$'s densities $\Rightarrow MTP_2 \ l(\mathbf{x})$, joint density for independent r.v.'s
- $\mathbf{X} \leq_{\mathbf{lr}} \mathbf{Y} \Rightarrow$ order preserved for all marginal distributions

# EXTENSION TO MULTIVARIATE CASE

- $\mathbf{X}$ and $\mathbf{Y}$ r.v.'s with pdf's $f$ and $g$, respectively
  - If either $\mathbf{X}$ or $\mathbf{Y}$ is $MTP_2$ then $\mathbf{X} \leq_{\mathbf{lr}} \mathbf{Y} \Leftrightarrow$
  - $g(\mathbf{x})f(\mathbf{y}) \leq g(\mathbf{y})f(\mathbf{x})$, $\forall \mathbf{x} \leq \mathbf{y}$, or, equivalently,
  - $g(\mathbf{x})/f(\mathbf{x})$ is increasing in the union of their supports
- $\mathbf{X} \leq_{\mathbf{lr}} \mathbf{Y} \Rightarrow \mathbf{g}(\mathbf{x})/\mathbf{f}(\mathbf{x})$ increasing in $\mathbf{x}$ (vice versa true only for $n = 1$)
- Equivalence between log-supermodular and $MTP_2$ functions
  - $f : \mathbb{R}^n \mapsto \mathbb{R}$ supermodular if $f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y})$
- A function $l : \mathbb{R}^n \mapsto \mathbb{R}^+$ with 2 continuous derivatives is $MTP_2$ if and only if

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln(l(\mathbf{x})) \geq 0, \; \forall i \neq j.$$

- $MTP_2$ positive dependence property
  - If $\mathbf{X}$ is $MTP_2$ then $Cov(\phi(\mathbf{X}), \psi(\mathbf{X})) \geq 0$ for $\phi$ and $\psi$ simultaneously monotone increasing or decreasing

# EXTENSION TO MULTIVARIATE CASE

- Multivariate prior $\boldsymbol{\pi}$ with pdf $\pi(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^n$
- Weight function $w : \mathbb{R}^n \mapsto \mathbb{R}^+$
- Weighted prior $\boldsymbol{\pi}_w(\boldsymbol{\theta}) = \dfrac{\omega(\boldsymbol{\theta})}{E^{\boldsymbol{\pi}}\left[\omega(\boldsymbol{\theta})\right]}\pi(\boldsymbol{\theta}), \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^n$
- Consider only increasing and decreasing weight functions for two reasons
  - generalization of convex and concave distortion functions in univariate case
  - $MTP_2$ prior $\boldsymbol{\pi}$ and increasing (decreasing) weight function $w \Rightarrow \boldsymbol{\pi} \leq_{lr} (\geq_{lr}) \boldsymbol{\pi}_w$

    $\Rightarrow$ weighted prior more (less) likely than prior to take on larger values
- Decision maker representing changes in prior $\boldsymbol{\pi}$ by weight functions: $w_1$ decreasing and $w_2$ increasing

  $\Rightarrow$ two weighted priors, $\boldsymbol{\pi}_{w_1}$ and $\boldsymbol{\pi}_{w_2}$, s.t. $\boldsymbol{\pi}_{w_1} \leq_{lr} \boldsymbol{\pi} \leq_{lr} \boldsymbol{\pi}_{w_2}$

# EXTENSION TO MULTIVARIATE CASE

- $\boldsymbol{\pi}$: $MTP_2$ prior

- $w_1$ and $w_2$: decreasing and increasing weight functions, respectively

- $\Rightarrow \Gamma_{w_1,w_2,\boldsymbol{\pi}} = \left\{\boldsymbol{\pi}' : \boldsymbol{\pi}_{w_1} \leq_{lr} \boldsymbol{\pi}' \leq_{lr} \boldsymbol{\pi}_{w_2}\right\}$: weighted band

- $\boldsymbol{\pi} \in \Gamma_{w_1,w_2,\boldsymbol{\pi}} \Rightarrow \Gamma_{w_1,w_2,\boldsymbol{\pi}}$ "neighborhood" of $\boldsymbol{\pi}$ s.t.

$$\begin{aligned}\Gamma_{w_1,w_2,\boldsymbol{\pi}} &\subseteq \left\{\boldsymbol{\pi}' : \boldsymbol{\pi}_{w_1} \leq_{st} \boldsymbol{\pi}' \leq_{st} \boldsymbol{\pi}_{w_2}\right\} \\ &\subseteq \left\{\boldsymbol{\pi}' : F_{\boldsymbol{\pi}_{w_1}}(\boldsymbol{\theta}) \geq F_{\boldsymbol{\pi}'}(\boldsymbol{\theta}) \geq F_{\boldsymbol{\pi}_{w_2}}(\boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}\right\}.\end{aligned}$$

- Likelihood ratio order does not apply, in general, when comparing two priors $\boldsymbol{\pi}'_1$ and $\boldsymbol{\pi}'_2$ in $\Gamma_{w_1,w_2,\pi}$ but each of them is just ordered w.r.t. $\boldsymbol{\pi}_{w_1}$ and $\boldsymbol{\pi}_{w_2}$

- Infinite number of priors in the class, e.g. all mixtures of two priors in the class (like $\boldsymbol{\pi}_{w_1}, \boldsymbol{\pi}_{w_2}$ and $\boldsymbol{\pi}$

# EXTENSION TO MULTIVARIATE CASE

- $\boldsymbol{\pi}$: $MTP_2$ prior
- $\Gamma_{w_1,w_2,\boldsymbol{\pi}}$: weighted band associated with $\boldsymbol{\pi}$ based on $w_1$ and $w_2$
- $\mathrm{x}$: observed data $\Rightarrow$ likelihood $l(\boldsymbol{\theta} \mid \mathrm{x})$
- $\boldsymbol{\pi}'_{\mathrm{x}}$: posterior w.r.t. prior $\boldsymbol{\pi}'$
- **IF** $l(\boldsymbol{\theta} \mid \mathrm{x})$ is $MTP_2$ in $\boldsymbol{\theta}$
- $\Rightarrow \boldsymbol{\pi}_{w_1,\mathrm{x}} \leq_{lr} \boldsymbol{\pi}'_{\mathrm{x}} \leq_{lr} \boldsymbol{\pi}_{w_2,\mathrm{x}}$ for all $\boldsymbol{\pi}' \in \Gamma_{w_1,w_2,\boldsymbol{\pi}}$

# EXTENSION TO MULTIVARIATE CASE

- Examples of weight functions
  - $w(\boldsymbol{\theta}) = \frac{1-F_\pi(\boldsymbol{\theta})}{\pi(\boldsymbol{\theta})}$
  - $w(\boldsymbol{\theta}) = \prod_{i=1}^n g_i(\theta_i)$ and $w(\boldsymbol{\theta}) = \sum_{i=1}^n g_i(\theta_i)$

    ($g_i$'s non-negative increasing (decreasing) functions $\Rightarrow$ $w$ increasing (decreasing))
  - $w(\boldsymbol{\theta}) = \prod_{i=1}^n \theta_i^{a_i-1}$ and $w(\boldsymbol{\theta}) = \sum_{i=1}^n \theta_i^{a_i-1}$ increasing (decreasing) when $a_i > 1$ ($a_i < 1$), for $i = 1, \ldots, n$
- Examples of $MTP_2$ likelihood
  - Sample from Pareto distribution
  - Sample from gamma distribution

# EXTENSION TO MULTIVARIATE CASE

- Failure data of door opening system of 40 underground trains
- Trains delivered to an European transportation company between 11/1989 and 3/1991 and put in service between 20/3/1990 to 20/7/1992
- Failure monitoring ended on 31/12/1998
- Odometer reading and failure date recorded upon failure, along with code of failed component (mechanical, electrical, etc.)
- Interest in
  - modelling failure history of electrical opening commands
  - predicting number of failures in future time intervals
  - checking reliability before warranty expiration

# EXTENSION TO MULTIVARIATE CASE

- $N(t)$: failures in the electrical opening system in the interval $(0, t]$
- $N(t)$: nonhomogeneous Poisson process (NHPP) with intensity function $\lambda(t)$ and increasing and invertible mean value function

$$m(t) = E[N(t)] = \int_0^t \lambda(x)dx,$$

  such that $m(\infty) = \infty$

- Power law process (PLP), with parameter $\boldsymbol{\theta} = (M, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$, with
  - intensity function $\lambda(t|\boldsymbol{\theta}) = M\beta t^{\beta-1}$
  - mean value function $m(t|\boldsymbol{\theta}) = Mt^{\beta}$
- $\mathbf{t}^* = (T_1, \ldots, T_n)$ observed failure times in $(0, T]$ s.t. $T_1 < \ldots < T_n$
- Likelihood function given by

$$l(\boldsymbol{\theta}|\mathbf{t}^*) = \prod_{i=1}^{n} \lambda(T_i) \cdot \exp(-m(T|\boldsymbol{\theta})) = \prod_{i=1}^{n} M\beta T_i^{\beta-1} \cdot \exp(-MT^{\beta})$$

# EXTENSION TO MULTIVARIATE CASE

- $\dfrac{\partial^2 \ln(l(\boldsymbol{\theta} \mid \mathbf{t}^*))}{\partial M \partial \beta} = -\ln(T)T^{\beta} \Rightarrow l(\boldsymbol{\theta} \mid \mathbf{t}^*) MTP_2$ in $\boldsymbol{\theta} \Leftrightarrow T \leq 1$

- $\Rightarrow$ Normalised times, i.e. divided by total time $T$

- Independent exponential priors: $M \sim Exp(\lambda)$ and $\beta \sim Exp(\mu)$

- Independent gamma posteriors:
  $M|data \sim Gamma(n+1, 1+\lambda)$ and $\beta|data \sim Gamma(n+1, \mu - \sum_{i=1}^{n} \ln(\frac{t_i}{T}))$

- Weighted band given by

$$\text{Lower: } \boldsymbol{\pi}_{\omega_1}(\boldsymbol{\theta}) \propto \lambda\mu M^{a-1}\beta^{b-1} \exp(-\lambda M)\exp(-\mu\beta)\exp[-cM\beta]$$

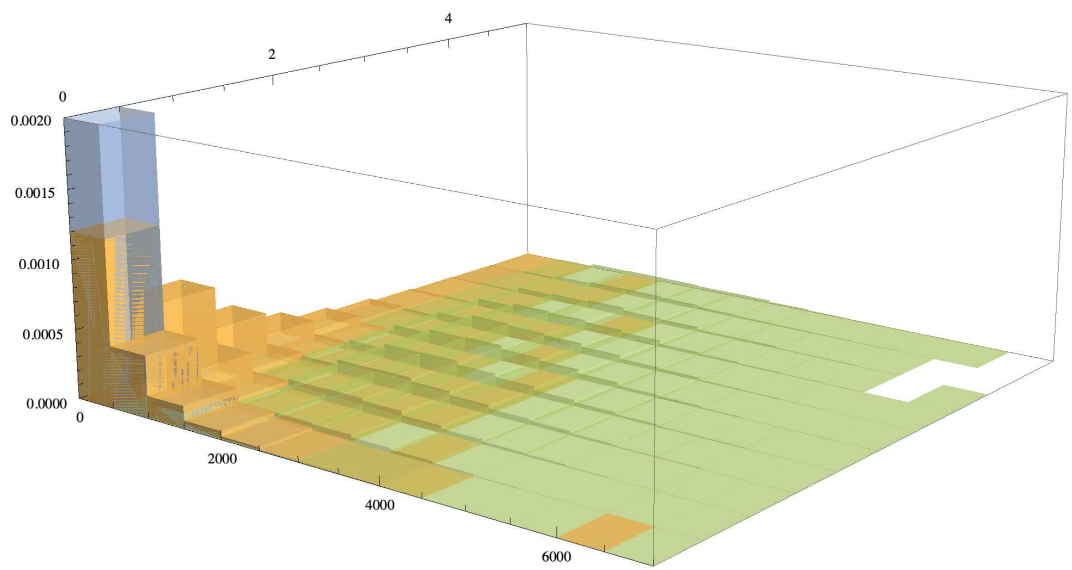
$$\text{Upper: } \boldsymbol{\pi}_{\omega_2}(\boldsymbol{\theta}) \propto \lambda\mu M^{a'-1}\beta^{b'-1} \exp(-\lambda M)\exp(-\mu\beta)(M^{c'} + \beta^{c'})$$

# EXTENSION TO MULTIVARIATE CASE

- $M \sim Exp(\lambda)$ and $\beta \sim Exp(\mu)$
  - Data from older trains (not used later) used to get numerically $M_0 = 495.5$ and $\beta_0 = 0.79$, comparing $m(t|\boldsymbol{\theta})$ and cumulative number of failures
  - $\lambda = 1/M_0$ and $\mu = 1/\beta_0$
- Hellinger distance between two densities $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$:
  $H(\mathbf{X}, \mathbf{Y}) = \frac{1}{2} \int_\Omega (\sqrt{f_{\mathbf{X}}(\mathrm{x})} - \sqrt{f_{\mathbf{Y}}(\mathrm{x})})^2 d\mathrm{x} = 1 - \int_\Omega \sqrt{f_{\mathbf{X}}(\mathrm{x}) f_{\mathbf{Y}}(\mathrm{x})} d\mathrm{x}$
- $(a = 0.8,\ b = 0.4,\ c = 0.17)$ and $(a' = 3.8,\ b' = 3.4,\ c' = 1.17)$ chosen to get a Hellinger distance of (approx.) 0.7 between starting prior and upper and lower ones

# EXTENSION TO MULTIVARIATE CASE

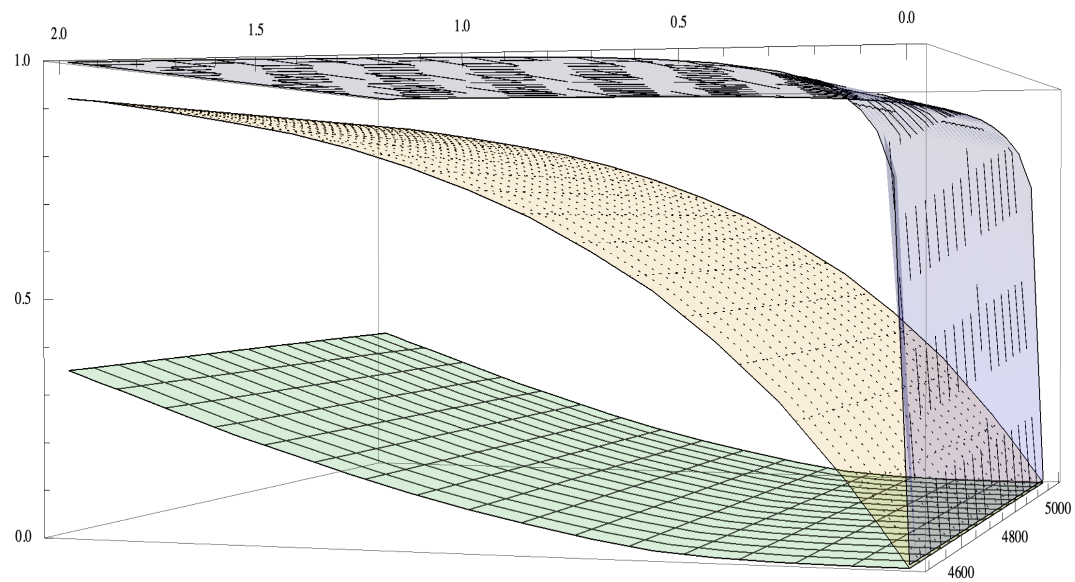
Prior distributions $\boldsymbol{\pi}_{w_1}$ (in blue), $\boldsymbol{\pi}$ (in orange) and $\boldsymbol{\pi}_{w_2}$ (in green)




Histograms for prior distributions

# EXTENSION TO MULTIVARIATE CASE

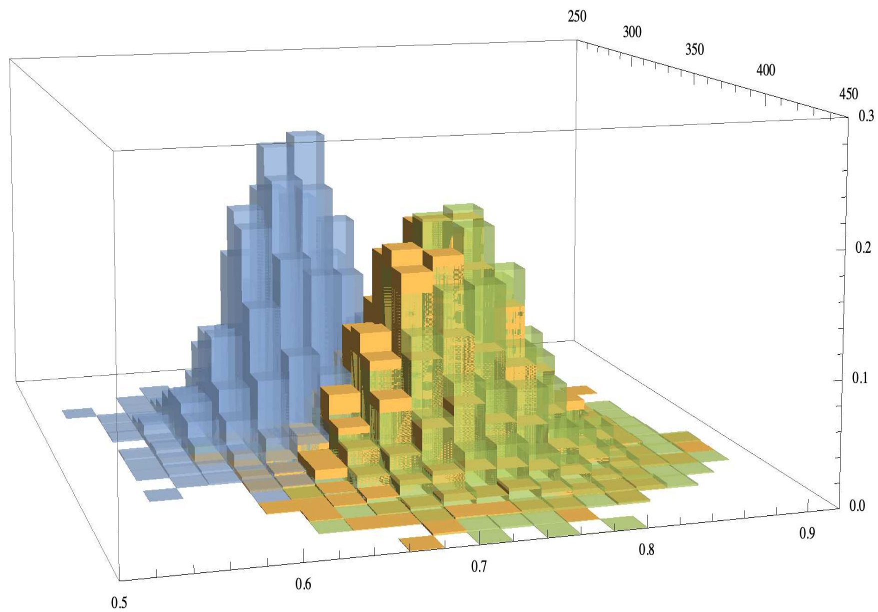
Prior distributions $\boldsymbol{\pi}_{w_1}$ (in blue), $\boldsymbol{\pi}$ (in orange) and $\boldsymbol{\pi}_{w_2}$ (in green)




Cdf's for prior distributions

# EXTENSION TO MULTIVARIATE CASE

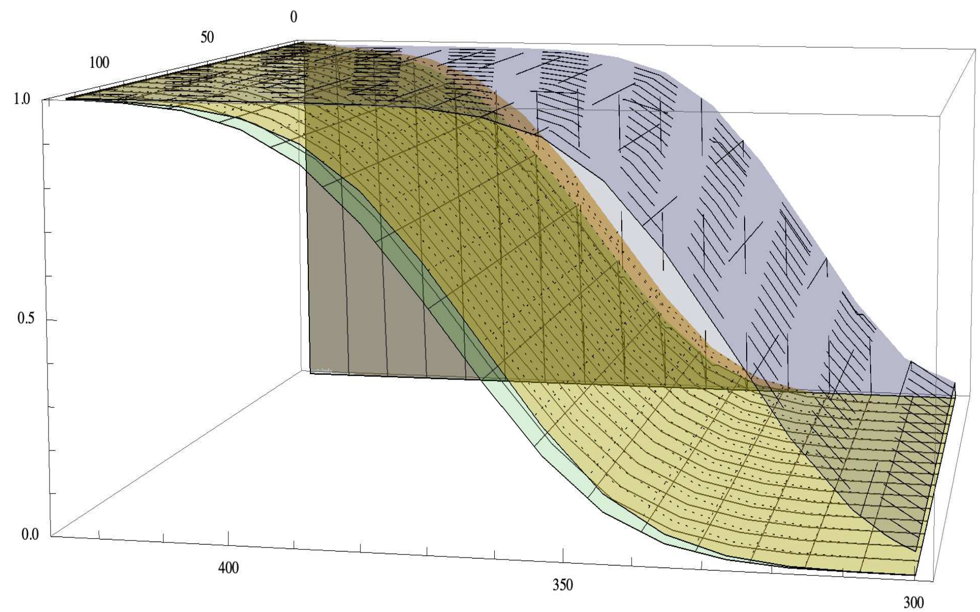
Prior distributions $\boldsymbol{\pi}_{w_1}$ (in blue), $\boldsymbol{\pi}$ (in orange) and $\boldsymbol{\pi}_{w_2}$ (in green)




Histograms for posterior distributions

# EXTENSION TO MULTIVARIATE CASE

Prior distributions $\boldsymbol{\pi}_{w_1}$ (in blue), $\boldsymbol{\pi}$ (in orange) and $\boldsymbol{\pi}_{w_2}$ (in green)




Cdf's for posterior distributions

# EXTENSION TO MULTIVARIATE CASE

| T | True value | 95% credibility Int. $\pi_{w_1,t}$ | Post. mean $\pi_{w_1,t}$ | 95% credibility Int. $\pi_t$ | Post. mean $\pi_t$ | 95% credibility Int. $\pi_{w_2,t}$ | Post. mean $\pi_{w_2,t}$ |
|---|---|---|---|---|---|---|---|
| 1992-1 | 83 | [47.63, 78.53] | 63 | [66.25, 101.95] | 84 | [70.95 , 107.79] | 89 |
| 1992-2 | 72 | [42.81, 72.30] | 57 | [63.20, 98.12] | 80 | [68.45 , 104.64] | 86 |
| 1992-3 | 62 | [39.83, 68.37] | 54 | [61.25, 95.66] | 78 | [66.84 , 102.62] | 84 |
| 1993-1 | 72 | [35.90, 63.29] | 49 | [48.22, 79.32] | 63 | [50.67 , 82.44] | 66 |
| 1993-2 | 62 | [35.52, 58.77] | 45 | [45.28, 75.51] | 60 | [47.82 , 78.78] | 63 |
| 1993-3 | 42 | [30.15, 55.58] | 42 | [43.16, 72.78] | 57 | [45.76 , 76.13] | 60 |
| 1994-1 | 62 | [31.59, 57.54] | 44 | [42.01, 71.31] | 56 | [43.53 , 73.29] | 58 |
| 1994-2 | 42 | [29.12, 54.22] | 41 | [39.77, 68.38] | 54 | [41.35 , 70.45] | 55 |
| 1994-3 | 35 | [27.28, 51.68] | 39 | [38.05, 66.12] | 52 | [39.66 , 68.24] | 53 |
| 1995-1 | 42 | [30.84, 56.54] | 43 | [40.92, 69.91] | 55 | [42.10 , 71.45] | 56 |
| 1995-2 | 35 | [28.92, 53.94] | 41 | [39.21, 67.66] | 53 | [40.42 , 69.26] | 54 |
| 1995-3 | 23 | [27.40, 51.86] | 39 | [37.82, 62.84] | 51 | [39.08 , 67.48] | 53 |
| 1996-1 | 35 | [26.20, 50.21] | 38 | [34.49, 61.42] | 47 | [35.35 , 62.58] | 48 |
| 1996-2 | 23 | [24.65, 48.05] | 36 | [32.99, 59.43] | 46 | [33.88 , 60.61] | 47 |
| 1997-1 | 23 | [22.75, 45.40] | 34 | [29.73, 55.04] | 42 | [30.46 , 56.03] | 43 |

Forecasts of the number of failures from 1993 to 1998

# REFERENCES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